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The Microgravity Isolation Mount: A Linearized State–Space Model á la Newton and Kane

R. David Hampton
University of Alabama in Huntsville, Huntsville, Alabama

Bjarni V. Tryggvason and Jean DeCarufel
Canadian Space Agency, St. Hubert, Quebec, Canada

Miles A. Townsend
University of Virginia, Charlottesville, Virginia

William O. Wagar
Lewis Research Center, Cleveland, Ohio

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Space Administration

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The Microgravity Isolation Mount:
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R. David Hampton[1]
University of Alabama in Huntsville
Huntsville, Alabama 35899

Bjarni V. Tryggvason[2] **and Jean DeCarufel**[3]
Canadian Space Agency
St. Hubert, Quebec, Canada J3Y8Y9

Miles A. Townsend[4]
University of Virginia
Charlottesville, Virginia 22903

William O. Wagar[5]
National Aeronautics and Space Administration
Lewis Research Center
21000 Brookpark Road, Mail Stop 500–216
Cleveland, Ohio 44135

Abstract

Vibration acceleration levels on large space platforms exceed the requirements of many space experiments. The Microgravity Vibration Isolation Mount (MIM) was built by the Canadian Space Agency to attenuate these disturbances to acceptable levels, and has been operational on the Russian Space Station Mir since May 1996. It has demonstrated good isolation performance and has supported several materials science experiments. The MIM uses Lorentz (voice-coil) magnetic actuators to levitate and isolate payloads at the individual experiment/sub-experiment (versus rack) level. Payload acceleration, relative position, and relative orientation (Euler-parameter) measurements are fed to a state-space controller. The controller, in turn, determines the actuator currents needed for effective experiment isolation. This paper presents the development of an algebraic, state-space model of the MIM, in a form suitable for optimal controller design. The equations are first derived using Newton's Second Law directly; then a second derivation (i.e., validation) of the same equations is provided, using Kane's approach.

[1] Associate Professor, Mechanical and Aerospace Engineering Department.
[2] Canadian Astronaut / MIM Principal Investigator and Payload Specialist.
[3] Control System Engineer.
[4] Wilson Professor, Department of Mechanical, Aerospace, and Nuclear Engineering.
[5] Senior Research Engineer.

Introduction

Acceleration measurements on the U.S. Space Shuttle and the Russian Mir Space Station show acceleration environments that are noisier than expected [1]. The acceleration environment on the International Space Station (ISS) likewise will not be as clean as originally anticipated; the ISS is unlikely to meet its microgravity requirements without the use of isolation systems [1], [2]. While the quasi-static acceleration levels due to such factors as atmospheric drag, gravity gradient, and spacecraft rotations are on the order of several micro-g, the vibration levels above 0.01 Hz are likely to exceed 300 micro-g rms, with peaks typically reaching milli-g levels [3]. These acceleration levels are sufficient to cause significant disturbances to many experiments that have fluid or vapor phases, including a large class of materials science experiments [4].

The Microgravity Vibration Isolation Mount (MIM) is designed to isolate experiments from the high frequency (>0.01 Hz) vibrations on the Space Shuttle, Mir, and ISS, while passing the quasi-static (<0.01 Hz) accelerations to the experiment [5]. It can provide up to 40 dB of acceleration attenuation to experiments of practically unlimited mass [6]. The acceleration-attenuation capability of the MIM is limited primarily by two factors: (1) the character of the umbilical required between the MIM base (stator) and the MIM experiment platform (flotor), and (2) the allowed stator-to-flotor rattlespace. A primary goal in MIM design was to isolate at the individual experiment, rather than entire rack, level; ideally the MIM isolates only the sensitive elements of an experiment. This typically results in a stator-to-flotor umbilical that can be greatly reduced in size and in the services it must provide. In the current implementation, the umbilical provides experiments with power, and data-acquisition and control services. Even with the approximately 70-wire umbilical the MIM has demonstrated good isolation performance [5].

The first MIM unit was launched in the Priroda laboratory module which docked with Mir in April 1996. The system has been operational on Mir since May 1996 and has supported several materials science experiments. An upgraded system (MIM-2) was flown on the U.S. Space Shuttle on mission STS-85 in August 1997 [5].

In order to design controllers for the MIM it was necessary to develop an appropriate dynamic model of the system. The present paper presents an algebraic, state-space model of the MIM, in a form appropriate for optimal controller design.

Problem Statement

The dynamic modeling and microgravity vibration isolation of a tethered, one-dimensional experiment platform was studied extensively by Hampton [7]. It was found that optimal control techniques could be effectively employed using a state-space system model, with relative-position, relative-velocity, and acceleration states. The experiment platform was assumed to be subject to Lorentz (voice-coil) electromagnetic actuation, and to indirect (umbilical-induced) and direct translational disturbances.

The task of the research presented below was to develop a corresponding state-space model of the MIM. Translational and rotational relative-position, relative-velocity, and acceleration states were to be included, with the rotational states employing Euler parameters and their derivatives. The MIM dynamic model must incorporate indirect and direct translational and rotational disturbances.

System Model

A schematic of the MIM is depicted in Figure 1. The stator, defined in reference frame S, is rigidly mounted to the orbiter. The flotor, frame F, is magnetically levitated above the stator by eight Lorentz actuators (two shown), each consisting of a flat racetrack-shaped electrical coil positioned between a set of Nd-Fe-Bo supermagnets. The coils and the supermagnets are fixed to the stator and flotor, respectively. Control currents passing through the coils interact with their respective supermagnet flux fields to produce control forces used for flotor isolation and disturbance attenuation [8].

The flotor has mass center F^* and a dextral coordinate system with unit vectors $\underline{\hat{f}}_1$, $\underline{\hat{f}}_2$, and $\underline{\hat{f}}_3$, and origin F_0. The stator (actually, stator-plus-orbiter) has mass center S^* and a dextral coordinate system with unit vectors $\underline{\hat{s}}_1$, $\underline{\hat{s}}_2$, and $\underline{\hat{s}}_3$, and origin S_0. The inertial reference frame Ⓝ is similarly defined by $\underline{\hat{n}}_1$, $\underline{\hat{n}}_2$, and $\underline{\hat{n}}_3$, and origin N_0. The umbilical is attached to the stator at S_u, and to the flotor at F_u. When the flotor is centered in its rattlespace (the “home” position), F^* and F_u are located at stator-fixed points F_h^*, and F_{uh}, respectively.

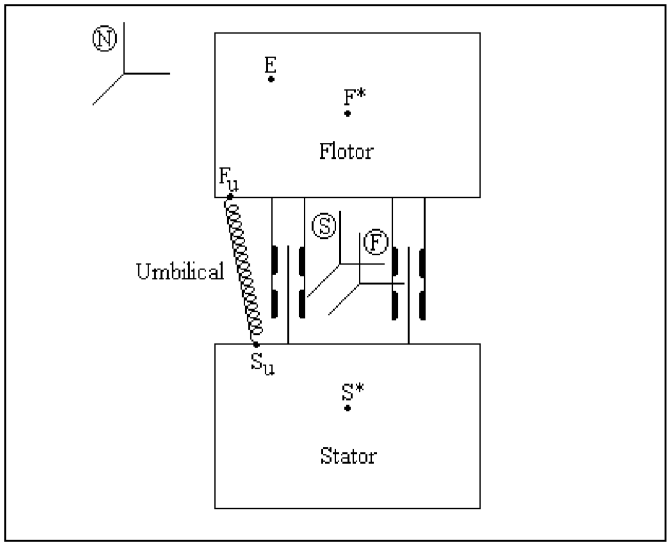

Figure 1. Schematic of the MIM

State Equations of Motion

Translational Equations of Motion

Let E be some flotor-fixed point of interest for which the acceleration is to be determined. If E has inertial position $\underline{r}_{N_0E}$, then its inertial velocity and acceleration are $\dot{\underline{r}}_{N\,E} = \frac{{}^N d}{dt}\left(\underline{r}_{N\,E}\right)$ and $\ddot{\underline{r}}_{N\,E} = \frac{{}^N d}{dt}\left[\frac{{}^N d}{dt}\left(\underline{r}_{N\,E}\right)\right]$, respectively. (The presuperscript indicates the reference frame of the differentiations. The subscripts indicate the vector origin and terminus.) The angular velocity and angular acceleration of the flotor with respect to the inertial frame are represented by ${}^N\underline{\omega}^F$ and ${}^N\underline{\alpha}^F$, respectively, where ${}^N\underline{\alpha}^F = \frac{{}^N d}{dt}\left({}^N\underline{\omega}^F\right)$.

Let $\underline{F}$ be the resultant of all external forces acting on the flotor; $\underline{M}^{F/F^*}$ (or simply $\underline{M}$), the moment resultant of these forces about F^*; m, the flotor mass; and $\underline{\underline{I}}^{F/F^*}$ (or $\underline{\underline{I}}$), the central inertia dyadic of the flotor for $\underline{\hat{f}}_1$, $\underline{\hat{f}}_2$, and $\underline{\hat{f}}_3$. Then Newton's Second Law for the flotor can be expressed in the following two forms:

$$\underline{F} = m\underline{\ddot{r}}_{N_0F^*} \qquad \text{(Eq. 1)}$$

and

$$\underline{M} = \underline{\underline{I}} \cdot {}^N\underline{\alpha}^F + {}^N\underline{\omega}^F \times \left(\underline{\underline{I}} \cdot {}^N\underline{\omega}^F\right). \qquad \text{(Eq. 2)}$$

From Equation (2),

$${}^N\underline{\alpha}^F = \underline{\underline{I}}^{-1} \cdot \left[\underline{M} - {}^N\underline{\omega}^F \times \left(\underline{\underline{I}} \cdot {}^N\underline{\omega}^F\right)\right]. \qquad \text{(Eq. 3)}$$

It will be useful to find an expression for $\underline{\ddot{r}}_{N_0E}$ in terms of the acceleration $\underline{\ddot{r}}_{N_0S_u}$ of the umbilical attachment point S_u, and in terms of the extension of the umbilical from its relaxed position.

Begin with the following: $\quad \underline{r}_{SF} = \underline{r}_{NE} + \underline{r}_{EF} - \underline{r}_{NS_0} - \underline{r}_{S_0S}$. $\qquad$ (Eq. 4)

Differentiation of Equation (4) yields

$$\underline{\dot{r}}_{SF} = \underline{\dot{r}}_{NE} + {}^N\underline{\omega}^F \times \underline{r}_{EF} - \underline{\dot{r}}_{NS_0} - {}^N\underline{\omega}^S \times \underline{r}_{S_0S} \ . \qquad \text{(Eq. 5)}$$

A second differentiation gives

$$\underline{\ddot{r}}_{SF} = \underline{\ddot{r}}_{N_0E} + {}^N\underline{\alpha}^F \times \underline{r}_{EF} + {}^N\underline{\omega}^F \times \left({}^N\underline{\omega}^F \times \underline{r}_{EF} \right) - \underline{\ddot{r}}_{NS} - {}^N\underline{\alpha}^S \times \underline{r}_{SS} - {}^N\underline{\omega}^S \times \left({}^N\underline{\omega}^S \times \underline{r}_{SS} \right). \qquad \text{(Eq. 6)}$$

Substitution for ${}^N\underline{\alpha}^F$ from Equation (3) into Equation (6) yields

$$\begin{aligned} \underline{\ddot{r}}_{S_uF_u} = \underline{\ddot{r}}_{N_0E} &+ \left\{\underline{\underline{I}}^{-1} \cdot \left[\underline{M} - {}^N\underline{\omega}^F \times \left(\underline{\underline{I}} \cdot {}^N\underline{\omega}^F\right)\right]\right\} \times \underline{r}_{EF} - \underline{\ddot{r}}_{NS} - {}^N\underline{\alpha}^S \times \underline{r}_{SS} \\ &- {}^N\underline{\omega}^S \times \left({}^N\underline{\omega}^S \times \underline{r}_{S_0S} \right) + {}^N\underline{\omega}^F \times \left({}^N\underline{\omega}^F \times \underline{r}_{EF} \right). \end{aligned} \qquad \text{(Eq. 7)}$$

In these equations

$${}^N\underline{\omega}^F = {}^N\underline{\omega}^S + {}^S\underline{\omega}^F \ . \qquad \text{(Eq. 8)}$$

Under the assumptions that ${}^N\underline{\omega}^S$ and ${}^N\underline{\alpha}^S$ are negligibly small and, therefore, that

$$\ddot{\underline{r}}_{N\ S} \approx \ddot{\underline{r}}_{N\ S_u}, \qquad \text{(Eq. 9)}$$

Equation (7) reduces to

$$\ddot{\underline{r}}_{S_uF_u} = \ddot{\underline{r}}_{N_0E} + \left\{ \underline{\underline{I}}^{-1} \cdot \left[\underline{M} - {}^S\underline{\omega}^F \times \left(\underline{\underline{I}} \cdot {}^S\underline{\omega}^F \right) \right] \right\} \times \underline{r}_{EF} - \ddot{\underline{r}}_{N_0S} + {}^S\underline{\omega}^F \times \left({}^S\underline{\omega}^F \times \underline{r}_{EF} \right). \qquad \text{(Eq. 10)}$$

Linearization about ${}^S\underline{\omega}^F = \underline{0}$ yields the following result:

$$\ddot{\underline{r}}_{S\ F} = \ddot{\underline{r}}_{N\ E} + \left\{ \underline{\underline{I}}^{-1} \cdot \underline{M} \right\} \times \underline{r}_{EF} - \ddot{\underline{r}}_{N\ S}\ . \qquad \text{(Eq. 11)}$$

Appropriate expressions for $\underline{F}$ and $\underline{M}$ will now be determined, for substitution into Equations (1) and (11), respectively. Those equations will be used in turn to obtain a more useful expression for $\ddot{\underline{r}}_{S_uF_u}$. [See Equations (43-48).]

The force resultant $\underline{F}$ is the vector sum of the eight actuator (coil) forces $\underline{F}_c^i\,(i = 1,\ldots,8)$, with resultant $\underline{F}_c$; of the umbilical force $\underline{F}_{ut}$, caused by umbilical extensions from the relaxed position; of the direct disturbance forces, with resultant $\underline{F}_d$; and of the gravitational force $\underline{F}_g$. Gravity may be neglected for a space vehicle in free-fall orbit. The moment resultant $\underline{M}$ is the vector sum of the moments due to the coil forces, with resultant $\underline{M}_c$; of the moment $\underline{M}_{ut}$ due to the umbilical force $\underline{F}_{ut}$; of the moment $\underline{M}_{ur}$ due to the umbilical rotations from the relaxed orientation; and of the moment $\underline{M}_d$ due to the direct disturbance forces. There is no moment due to gravity, since $\underline{M}$ is about the flotor center of mass F^*. In equation form, assuming the i^{th} coil force to be applied at flotor-fixed point B_i,

$$\underline{F} = \sum_{i=1}^{8} \underline{F}_c^i + \underline{F}_{ut} + \underline{F}_d \qquad \text{(Eq. 12)}$$

and

$$\underline{M} = \sum_{i=1}^{8} \underline{r}_{F\ B} \times \underline{F}_c^i + \underline{r}_{F\ F} \times \underline{F}_{ut} + \underline{M}_{ur} + \underline{M}_d\ . \qquad \text{(Eq. 13)}$$

More explicit expressions for $\underline{F}_c^i$ and $\underline{F}_{ut}$ will now be developed. If the actuator has coil current $I_i \hat{\underline{I}}_i$, length L_i, and magnetic flux density $B_i \hat{\underline{B}}_i$, then the associated actuator force becomes

$$\underline{F}_c^i = -I_i L_i B_i \hat{\underline{I}}_i \times \hat{\underline{B}}_i . \tag{Eq. 14}$$

Assume a translational stiffness K_t^i for an umbilical elongation in the $\hat{\underline{s}}_i$ direction, and a corresponding translational damping C_t^i. Let $\underline{F}_b$ represent the umbilical bias force, exerted by the umbilical on the flotor in the home position. Then the total force of the umbilical on the flotor becomes

$$\underline{F}_{ut} = -\left\{ \sum_{i=1}^{3} K_t^i \left[\left(\underline{r}_{S\ F} - \underline{r}_{S\ F} \right) \cdot \hat{\underline{s}}_i \right] \hat{\underline{s}}_i + \sum_{i=1}^{3} C_t^i \left[\frac{{}^N d}{dt} \left(\underline{r}_{S\ F} - \underline{r}_{S\ F} \right) \cdot \hat{\underline{s}}_i \right] \hat{\underline{s}}_i \right\} + \underline{F}_b . \tag{Eq. 15}$$

Define the following, for $i = 1, 2, 3$:

$$x_{ai} = \left(\underline{r}_{S\ F} - \underline{r}_{S\ F} \right) \cdot \hat{\underline{s}}_i \tag{Eq. 16}$$

and

$$x_{bi} = \dot{x}_{ai} . \tag{Eq. 17}$$

If ${}^N\underline{\omega}^S \approx \underline{0}$, Equation (15) becomes

$$\underline{F}_{ut} = -\left[\sum_{i=1}^{3} \left(K_t^i x_{ai} + C_t^i x_{bi} \right) \hat{\underline{s}}_i \right] + \underline{F}_b . \tag{Eq. 18}$$

The relative positions x_{ai} and the relative velocities x_{bi} will be six of the nine translational states used in the state-space formulation of the system equations of motion.

As with $\underline{F}_c^i$ and $\underline{F}_{ut}$ above, $\underline{M}_{ur}$ can also be expressed in more explicit form, in analogous fashion. Assume a rotational stiffness K_r^i and a rotational damping C_r^i, for umbilical twist about the $\hat{\underline{s}}_i$ direction. Let $\phi^{F/S} \hat{\underline{n}}_\phi$ represent the rotation of the flotor, relative to the stator, from the relative position in which the $\underline{\hat{f}}_i$ and $\underline{\hat{s}}_i$ coordinate systems are aligned. ${}^{F/S}\underline{\hat{n}}_\phi$ is the rotation axis, and ϕ is the angle of twist about that axis. Let $\underline{M}_b$ represent the umbilical bias moment, exerted by the umbilical on the flotor in the home position. Then the moment $\underline{M}_{ur}$ can be expressed by the following:

$$\underline{M}_{ur} = -\left\{ \sum_{i=1}^{3} K_r^i \left[\left(\phi^{F/S} \underline{\hat{n}}_\phi \right) \cdot \underline{\hat{s}}_i \right] \underline{\hat{s}}_i + \sum_{i=1}^{3} C_r^i \left[\frac{{}^N d}{dt} \left(\phi^{F/S} \underline{\hat{n}}_\phi \right) \cdot \underline{\hat{s}}_i \right] \underline{\hat{s}}_i \right\} + \underline{M}_b, \qquad \text{(Eq. 19)}$$

or

$$\underline{M}_{ur} = -\left\{ \sum_{i=1}^{3} K_r^i \left[\left(\phi^{F/S} \underline{\hat{n}}_\phi \right) \cdot \underline{\hat{s}}_i \right] \underline{\hat{s}}_i + \sum_{i=1}^{3} C_r^i \left[\left(\dot{\phi}^{F/S} \underline{\hat{n}}_\phi + \phi^{F/S} \dot{\underline{\hat{n}}}_\phi \right) \cdot \underline{\hat{s}}_i \right] \underline{\hat{s}}_i \right\} + \underline{M}_b. \qquad \text{(Eq. 20)}$$

Equation (20) can be expressed in alternate form using Euler parameters. Let ${}^{F/S}\underline{\hat{n}}_\phi$ be described in Ⓢ by

$${}^{F/S}\underline{\hat{n}}_\phi = e_1 \underline{\hat{s}}_1 + e_2 \underline{\hat{s}}_2 + e_3 \underline{\hat{s}}_3. \qquad \text{(Eq. 21)}$$

Define the following Euler parameters [9].

$${}^{F/S}\beta_0 = \cos\frac{\phi}{2}, \qquad \text{(Eq. 22)}$$

$${}^{(S)F/S}\beta_1 = e_1 \sin\frac{\phi}{2}, \qquad \text{(Eq. 23)}$$

$${}^{(S)F/S}\beta_2 = e_2 \sin\frac{\phi}{2}, \qquad \text{(Eq. 24)}$$

$${}^{(S)F/S}\beta_3 = e_3 \sin\frac{\phi}{2}, \qquad \text{(Eq. 25)}$$

and

$${}^{F/S}\underline{\beta} = \sin\frac{\phi}{2} \, {}^{F/S}\underline{\hat{n}}_\phi. \qquad \text{(Eq. 26)}$$

For small values of ϕ, the Euler parameters can be simplified:

$${}^{F/S}\beta_0 \approx 1, \qquad \text{(Eq. 27)}$$

$$ {}^{(S)F/S}\beta_1 = e_1\,\phi/2\,, \qquad \text{(Eq. 28)} $$

$$ {}^{(S)F/S}\beta_2 = e_2\,\phi/2\,, \qquad \text{(Eq. 29)} $$

$$ {}^{(S)F/S}\beta_3 = e_3\,\phi/2\,, \qquad \text{(Eq. 30)} $$

and

$$ {}^{F/S}\underline{\beta} = \frac{\phi}{2}\,{}^{F/S}\underline{\hat{n}}_{\phi}\,. \qquad \text{(Eq. 31)} $$

Note that, for small angles,

$$ \phi\,{}^{F/S}\underline{\hat{n}}_{\phi} = 2\,{}^{F/S}\underline{\beta}\,. \qquad \text{(Eq. 32)} $$

This equation can be used to simplify the stiffness terms of Equation (20).

As for the damping term, Equation (31) can be differentiated to yield

$$ {}^{F/S}\underline{\dot{\beta}} = \frac{\dot{\phi}}{2}\cos\frac{\phi}{2}\,{}^{F/S}\underline{\hat{n}}_{\phi} + \sin\frac{\phi}{2}\,{}^{F/S}\underline{\dot{\hat{n}}}_{\phi}\,, \qquad \text{(Eq. 33)} $$

or, for small angles,

$$ 2\,{}^{F/S}\underline{\dot{\beta}} = \dot{\phi}\,{}^{F/S}\underline{\hat{n}}_{\phi} + \phi\,{}^{F/S}\underline{\dot{\hat{n}}}_{\phi}\,. \qquad \text{(Eq. 34)} $$

Equation (20) now becomes

$$ \underline{M}_{ur} = -2\left[\sum_{i=1}^{3} K_r^i\left({}^{F/S}\underline{\beta}\cdot\underline{\hat{s}}_i\right)\underline{\hat{s}}_i + \sum_{i=1}^{3} C_r^i\left({}^{F/S}\underline{\dot{\beta}}\cdot\underline{\hat{s}}_i\right)\underline{\hat{s}}_i\right] + \underline{M}_b\,. \qquad \text{(Eq. 35)} $$

Define the following, for $i = 1, 2, 3$:

$$ x_{di} = {}^{F/S}\underline{\beta}\cdot\underline{\hat{s}}_i \qquad \text{(Eq. 36)} $$

and

$$ x_{ei} = \dot{x}_{di}\,. \qquad \text{(Eq. 37)} $$

The assumption that ${}^{N}\underline{\omega}^{S}$ is negligible yields, finally,

$$ \underline{M}_{ur} = -2\left\{\sum_{i=1}^{3}\left[K_r^i x_{di} + C_r^i x_{ei}\right]\underline{\hat{s}}_i\right\} + \underline{M}_b\,. \qquad \text{(Eq. 38)} $$

Note that Equation (11) describes $\ddot{\underline{r}}_{S_u F_u}$ in terms of the acceleration of an arbitrary flotor-fixed point E. For E located at flotor mass center F^*, Equation (11) can be used straightforwardly with Equation (1) to yield

$$\ddot{\underline{r}}_{S\ F} = \frac{1}{m}\underline{F} + \left(\underline{\underline{I}}^{-1} \cdot \underline{M}\right) \times \underline{r}_{F^*F} - \ddot{\underline{r}}_{N_0 S} \; . \qquad \text{(Eq. 39)}$$

Define now three unknown-acceleration terms, to be used with Equation (39). The first term represents the indirect translational acceleration disturbance input to the flotor, applied at the stator end of the umbilical:

$$\underline{a}_{in} = \ddot{\underline{r}}_{N_0 S_u} \; . \qquad \text{(Eq. 40)}$$

The second term represents the direct translational acceleration disturbance to the flotor, due to unknown disturbance force $\underline{F}_d$:

$$\underline{a}_d = \frac{1}{m}\underline{F}_d \; . \qquad \text{(Eq. 41)}$$

And the third represents the direct angular acceleration disturbance input to the flotor, due to $\underline{F}_d$:

$$\underline{\alpha}_d = \underline{\underline{I}}^{-1} \cdot \underline{M}_d \, , \qquad \text{(Eq. 42)}$$

Substitution from Equations (12), (13), (14), (18), (38), (40), (41), and (42) into (39) yields the following result:

$$\ddot{\underline{r}}_{S_u F_u} = \frac{1}{m}\left(\underline{F}_c + \underline{F}_{ut}\right) + \underline{\underline{I}}^{-1} \cdot \left(\underline{M}_c + \underline{M}_{ut} + \underline{M}_{ur}\right) \times \underline{r}_{F\ F} + \underline{\alpha}_d \times \underline{r}_{F\ F} - \underline{a}_{in} + \underline{a}_d \, , \qquad \text{(Eq. 43)}$$

where

$$\underline{F}_c = \sum_{i=1}^{8} \underline{F}_c^i = \sum_{i=1}^{8} \left(-I_i L_i B_i \underline{\hat{I}}_i \times \underline{\hat{B}}_i\right), \qquad \text{(Eq. 44)}$$

$$\underline{F}_{ut} = -\left[\sum_{i=1}^{3} \left(K_t^i x_{ai} + C_t^i x_{bi}\right) \underline{\hat{s}}_i\right] + \underline{F}_b \, , \qquad \text{(Eq. 45)}$$

$$\underline{M}_c = \sum_{i=1}^{8} \underline{M}_c^i = \sum_{i=1}^{8} \underline{r}_{F^* B_i} \times \left(-I_i L_i B_i \underline{\hat{I}}_i \times \underline{\hat{B}}_i\right), \qquad \text{(Eq. 46)}$$

$$\underline{M}_{ut} = -\underline{r}_{F^*F_u} \times \left[\sum_{i=1}^{3} \left(K_t^i x_{ai} + C_t^i x_{bi} \right) \hat{\underline{s}}_i \right], \quad \text{(Eq. 47)}$$

and
$$\underline{M}_{ur} = -2 \left\{ \sum_{i=1}^{3} \left[K_r^i x_{di} + C_r^i x_{ei} \right] \hat{\underline{s}}_i \right\} + \underline{M}_b. \quad \text{(Eq. 48)}$$

Substitution from Equation (43) into Equation (11) produces the following equation for the acceleration of arbitrary flotor point E:

$$\ddot{\underline{r}}_{N_0 E} = \frac{1}{m}\left(\underline{F}_c + \underline{F}_{ut}\right) + \underline{\underline{I}}^{-1} \cdot \left(\underline{M}_c + \underline{M}_{ut} + \underline{M}_{ur}\right) \times \underline{r}_{F\,E} + \underline{\alpha}_d \times \underline{r}_{F\,E} + \underline{a}_d. \quad \text{(Eq. 49)}$$

Assuming ${}^{N}\underline{\omega}^{S}$ to be negligible, one also has the following:

$$\dot{\underline{r}}_{S\,F} = \frac{{}^{S}d}{dt}\left(\underline{r}_{S\,F}\right), \quad \text{(Eq. 50)}$$

and
$$\ddot{\underline{r}}_{S\,F} = \frac{{}^{S}d^2}{dt^2}\left(\underline{r}_{S\,F}\right). \quad \text{(Eq. 51)}$$

(Note that assuming ${}^{N}\underline{\omega}^{S}$ to be negligible does not imply that Ⓢ and Ⓝ are identical; it means rather that Ⓢ can be treated as if it is in pure translation relative to Ⓝ for the frequencies of interest.) Equations (43), (49), (50), and (51) provide the basis for a state-space form of the translational equations of motion, using x_{ai}, x_{bi}, and low-pass-filtered approximations to the $\hat{\underline{s}}_i$ components of $\ddot{\underline{r}}_{N_0 E}$ [see Equations (94) and (99)], as states.

Rotational Equations of Motion

Let ${}^{F/S}\underline{\beta} = \sin\frac{\phi}{2}\,{}^{F/S}\hat{\underline{n}}_{\phi}$ as before [Eq. (26)]. Differentiating the left side twice produces

$$\frac{{}^{N}d}{dt}\left({}^{F/S}\underline{\beta}\right) = \frac{{}^{S}d}{dt}\left({}^{F/S}\underline{\beta}\right) + {}^{N}\underline{\omega}^{S} \times {}^{F/S}\underline{\beta} \quad \text{(Eq. 52)}$$

and
$$\frac{{}^{N}d^2}{dt^2}\left({}^{F/S}\underline{\beta}\right) = \frac{{}^{S}d^2}{dt^2}\left({}^{F/S}\underline{\beta}\right) + {}^{N}\underline{\omega}^{S} \times \frac{{}^{S}d}{dt}\left({}^{F/S}\underline{\beta}\right) + {}^{N}\dot{\underline{\omega}}^{S} \times {}^{F/S}\underline{\beta} + {}^{N}\underline{\omega}^{S} \times {}^{F/S}\dot{\underline{\beta}}. \quad \text{(Eq. 53)}$$

Assuming as before that ${}^{N}\underline{\omega}^{S} \approx \underline{0}$, Equations (52) and (53) become, respectively,

$$ {}^{F/S}\dot{\underline{\beta}} = \frac{{}^{S}d}{dt}\left({}^{F/S}\underline{\beta} \right) \qquad \text{(Eq. 54)} $$

and

$$ {}^{F/S}\ddot{\underline{\beta}} = \frac{{}^{S}d^{2}}{dt^{2}}\left({}^{F/S}\underline{\beta} \right). \qquad \text{(Eq. 55)} $$

Returning to Equation (26), two differentiations of the right side yield

$$ \frac{{}^{N}d^{2}}{dt^{2}}\left({}^{F/S}\underline{\beta} \right) = \frac{\ddot{\phi}}{2}\cos\frac{\phi}{2}\,{}^{F/S}\underline{\hat{n}}_{\phi} - \left(\frac{\dot{\phi}}{2}\right)^{2}\sin\frac{\phi}{2}\,{}^{F/S}\underline{\hat{n}}_{\phi} + \dot{\phi}\cos\frac{\phi}{2}\,{}^{F/S}\dot{\underline{\hat{n}}}_{\phi} + \sin\frac{\phi}{2}\,{}^{F/S}\ddot{\underline{\hat{n}}}_{\phi}. \qquad \text{(Eq. 56)} $$

Linearizing about $\phi = 0$ and $\dot{\phi} = 0$, Equation (56) becomes

$$ 2\,{}^{F/S}\ddot{\underline{\beta}} = \ddot{\phi}\,{}^{F/S}\underline{\hat{n}}_{\phi} + 2\dot{\phi}\,{}^{F/S}\dot{\underline{\hat{n}}}_{\phi} + \phi\,{}^{F/S}\ddot{\underline{\hat{n}}}_{\phi}. \qquad \text{(Eq. 57)} $$

Equations (34), (54), (55), and (57) provide the basis for a state-space form of the rotational equations of motion, using as states the $\underline{\hat{s}}_i$ components of ${}^{F/S}\underline{\beta}$ and of ${}^{F/S}\dot{\underline{\beta}}$ (i.e., x_{di} and x_{ei}, respectively, for $i = 1, 2, 3$).

Equations of Motion in State-Space Form

From Equation (16),

$$ \underline{r}_{S\ F} - \underline{r}_{S\ F} = x_{a1}\underline{\hat{s}}_1 + x_{a2}\underline{\hat{s}}_2 + x_{a3}\underline{\hat{s}}_3. \qquad \text{(Eq. 58)} $$

Differentiation, along with the use of Equations (17) and (50), leads to the following:

$$ \dot{\underline{r}}_{S\ F} - \dot{\underline{r}}_{S\ F} = x_{b1}\underline{\hat{s}}_1 + x_{b2}\underline{\hat{s}}_2 + x_{b3}\underline{\hat{s}}_3. \qquad \text{(Eq. 59)} $$

A second differentiation yields

$$ \ddot{\underline{r}}_{S\ F} - \ddot{\underline{r}}_{S\ F} = \dot{x}_{b1}\underline{\hat{s}}_1 + \dot{x}_{b2}\underline{\hat{s}}_2 + \dot{x}_{b3}\underline{\hat{s}}_3. \qquad \text{(Eq. 60)} $$

Introduce the use of a presuperscript in parentheses to indicate the coordinate system used for componentiation. (This notation allows vectors to be expressed unambiguously in terms of their measure numbers.) Then Equations (58) and (59) take the respective forms,

$$^{(S)}\underline{r}_{S\,F} - {}^{(S)}\underline{r}_{S\,F} = \begin{Bmatrix} x_{a1} \\ x_{a2} \\ x_{a3} \end{Bmatrix} = \underline{x}_a \tag{Eq. 61}$$

and

$$^{(S)}\underline{\dot{r}}_{S\,F} - {}^{(S)}\underline{\dot{r}}_{S\,F} = {}^{(S)}\underline{\dot{r}}_{S\,F} = \underline{\dot{x}}_a = \underline{x}_b, \tag{Eq. 62}$$

where $\underline{x}_a$ and $\underline{x}_b$ are defined as indicated. $\underline{x}_d$ and $\underline{x}_e$ have corresponding definitions. [Cf. Equations (36) and (37).]

Equations (43) and (60) can be used together to develop a state-space equation for $\underline{\dot{x}}_b$. First, express Equation (60) in measure-number form:

$$^{(S)}\underline{\ddot{r}}_{S_u F_u} = \begin{Bmatrix} \dot{x}_{b1} \\ \dot{x}_{b2} \\ \dot{x}_{b3} \end{Bmatrix} = \underline{\dot{x}}_b \,. \tag{Eq. 63}$$

Next, define rotation matrix $^{S/F}Q$ by

$$\begin{Bmatrix} \underline{\hat{s}}_1 \\ \underline{\hat{s}}_2 \\ \underline{\hat{s}}_3 \end{Bmatrix} = {}^{S/F}Q \begin{Bmatrix} \underline{\hat{f}}_1 \\ \underline{\hat{f}}_2 \\ \underline{\hat{f}}_3 \end{Bmatrix}, \tag{Eq. 64}$$

where the prefix indicates the rotation of frame S relative to frame F . Finally, observe that, for arbitrary vectors

$$\underline{r}_1 = x_1 \underline{\hat{f}}_1 + y_1 \underline{\hat{f}}_2 + z_1 \underline{\hat{f}}_3 \tag{Eq. 65}$$

and

$$\underline{r}_2 = x_2 \underline{\hat{f}}_1 + y_2 \underline{\hat{f}}_2 + z_2 \underline{\hat{f}}_3, \tag{Eq. 66}$$

the cross product can be expressed in determinant form by

$$\underline{r}_1 \times \underline{r}_2 = \begin{vmatrix} \underline{\hat{f}}_1 & \underline{\hat{f}}_2 & \underline{\hat{f}}_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}, \tag{Eq. 67}$$

or in matrix form (i.e., using measure numbers) [9], by

$$^{(F)}\left(\underline{r}_1 \times \underline{r}_2\right) = \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix}. \qquad \text{(Eq. 68)}$$

Represent the above skew-symmetric matrix by $^{(F)}\underline{r}_1^{\times}$. Using this notation, Equation (43) can be expressed as follows:

$$\underline{\dot{x}}_b = \frac{1}{m}\left[{}^{(S)}\underline{F}_c + {}^{(S)}\underline{F}_{ut}\right] - {}^{S/F}Q\ {}^{(F)}\underline{r}^{\times}_{F^*F_u} I^{-1}\left[{}^{(F)}\underline{M}_c + {}^{(F)}\underline{M}_{ut} + {}^{(F)}\underline{M}_{ur}\right] - {}^{S/F}Q\ {}^{(F)}\underline{r}^{\times}_{F^*F_u} I^{-1}\ {}^{(F)}\underline{M}_d - {}^{(S)}\underline{a}_{in} + {}^{(S)}\underline{a}_d, \qquad \text{(Eq. 69)}$$

where I is the inertia matrix corresponding to $\underline{\underline{I}}$.

Linearizing Equation (3) about ${}^{N}\underline{\omega}^{F} = \underline{0}$ yields

$${}^{N}\underline{\alpha}^{F} = \underline{\underline{I}}^{-1} \cdot \underline{M}. \qquad \text{(Eq. 70)}$$

But

$${}^{N}\underline{\alpha}^{F} = \frac{{}^{N}d^2}{dt^2}\left(\phi^{F/S}\underline{\hat{n}}_\phi\right) = \ddot{\phi}^{F/S}\underline{\hat{n}}_\phi + 2\dot{\phi}^{F/S}\underline{\dot{\hat{n}}}_\phi + \phi^{F/S}\underline{\ddot{\hat{n}}}_\phi. \qquad \text{(Eq. 71)}$$

From Equations (57), (70), and (71),

$$2\ {}^{F/S}\underline{\ddot{\beta}} = \underline{\underline{I}}^{-1} \cdot \underline{M}, \qquad \text{(Eq. 72)}$$

or, equivalently,

$$2\left[\frac{{}^{N}d^2}{dt^2}\left({}^{F/S}\underline{\beta}\right)\right] = \underline{\underline{I}}^{-1} \cdot \underline{M}. \qquad \text{(Eq. 73)}$$

Application of Equation (55) leads directly to

$$\frac{{}^{S}d^2}{dt^2}\left({}^{F/S}\underline{\beta}\right) = \frac{1}{2}\underline{\underline{I}}^{-1} \cdot \underline{M}. \qquad \text{(Eq. 74)}$$

In measure-number form,

$$\underline{\dot{x}}_e = \frac{1}{2}\ {}^{S/F}Q\, I^{-1}\ {}^{(F)}\underline{M}, \qquad \text{(Eq. 75)}$$

or, equivalently,

$$\underline{\dot{x}}_e = \frac{1}{2}\ {}^{S/F}Q\, I^{-1}\left[{}^{(F)}\underline{M}_{ut} + {}^{(F)}\underline{M}_{ur} + {}^{(F)}\underline{M}_c\right] + \frac{1}{2}\ {}^{S/F}Q\ {}^{(F)}\underline{\alpha}_d. \qquad \text{(Eq. 76)}$$

Six state equations of the system are given by Equations (17) and (37), iterating on *i*; six more, by Equations (69) and (76). The latter six are written in terms of the various forces and moments acting on the system, which loads have been defined in vector form by Equations (44) through (48). These loads can be rewritten in measure-number form and substituted into Equations (69) and (76), as follows. Beginning with Equation (44), the i^{th} control force can be expressed as

$$^{(S)}\underline{F}_c^i = \left[-L_i \; ^{(S)}\underline{\hat{I}}_i^{\times} \; ^{S/F}Q \; B_i \; ^{(F)}\underline{\hat{B}}_i\right] I_i = F_c^i u_i , \qquad \text{(Eq. 77)}$$

The resultant control force becomes

$$^{(S)}\underline{F}_c = \sum_{i=1}^{8} {}^{(S)}\underline{F}_c^i = F_c \underline{u} , \qquad \text{(Eq. 78)}$$

where F_c^i, F_c, u_i, and $\underline{u}$ are defined as indicated.

Next, using Equation (61) with (45), the translational force the umbilical exerts on the flotor can be expressed by

$$^{(S)}\underline{F}_{ut} = -K_t \underline{x}_a - C_t \underline{x}_b + {}^{(S)}\underline{F}_b = F_{uta} \underline{x}_a + F_{utb} \underline{x}_b + {}^{(S)}\underline{F}_b , \qquad \text{(Eq. 79)}$$

where

$$K_t = \begin{bmatrix} K_t^1 & 0 & 0 \\ 0 & K_t^2 & 0 \\ 0 & 0 & K_t^3 \end{bmatrix}, \qquad \text{(Eq. 80)}$$

$$C_t = \begin{bmatrix} C_t^1 & 0 & 0 \\ 0 & C_t^2 & 0 \\ 0 & 0 & C_t^3 \end{bmatrix}, \qquad \text{(Eq. 81)}$$

and F_{uta} and F_{utb} are defined as indicated.

The i^{th} control force $\underline{F}_c^i$ exerts on the flotor a moment $\underline{M}_c^i$, defined by Equation (46). Using again the notation introduced with Equation (68), this moment can be expressed by

$$^{(F)}\underline{M}_c^i = \left[-L_i \; B_i \; ^{(F)}\underline{r}_{F^*B_i}^{\times} \; ^{S/F}Q^T \; ^{(S)}\underline{\hat{I}}_i^{\times} \; ^{S/F}Q \; ^{(F)}\underline{\hat{B}}_i\right] I_i = M_c^i u_i , \qquad \text{(Eq. 82)}$$

and the resultant moment by

$$ {}^{(F)}\underline{M}_c = \sum_{i=1}^{8} {}^{(F)}\underline{M}_c^i = M_c\,\underline{u}, \qquad \text{(Eq. 83)} $$

where M_c^i and M_c are defined as indicated.

The umbilical force $\underline{F}_{ut}$ exerts on the flotor a moment $\underline{M}_{ut}$, given by Equation (47). Substituting from Equation (79), this moment can be expressed by

$$ {}^{(F)}\underline{M}_{ut} = {}^{(F)}\underline{r}^{\times}_{F^*F_u}\;{}^{S/F}Q^T\left[F_{uta}\underline{x}_a + F_{utb}\underline{x}_b\right]; \qquad \text{(Eq. 84)} $$

or, alternatively,

$$ {}^{(F)}\underline{M}_{ut} = M_{uta}\underline{x}_a + M_{utb}\underline{x}_b, \qquad \text{(Eq. 85)} $$

for M_{uta} and M_{utb} appropriately defined.

Finally, Equation (48) expresses the moment $\underline{M}_{ur}$ that the umbilical applies to the flotor due to umbilical rotational stiffness. The following equations express $\underline{M}_{ur}$ in measure-number form:

$$ {}^{(F)}\underline{M}_{ur} = -2\,{}^{S/F}Q^T K_r\underline{x}_d - 2\,{}^{S/F}Q^T C_r\underline{x}_e + {}^{(F)}\underline{M}_b = M_{urd}\underline{x}_d + M_{ure}\underline{x}_e + {}^{(F)}\underline{M}_b, \qquad \text{(Eq. 86)} $$

where

$$ K_r = \begin{bmatrix} K_r^1 & 0 & 0 \\ 0 & K_r^2 & 0 \\ 0 & 0 & K_r^3 \end{bmatrix}, \qquad \text{(Eq. 87)} $$

$$ C_r = \begin{bmatrix} C_r^1 & 0 & 0 \\ 0 & C_r^2 & 0 \\ 0 & 0 & C_r^3 \end{bmatrix}, \qquad \text{(Eq. 88)} $$

and M_{urd} and M_{ure} are appropriately defined.

Substituting from Equations (77) through (88), Equations (69) and (76) become, respectively,

$$ \begin{aligned} \dot{\underline{x}}_b =& \left(\frac{1}{m}F_{uta} - {}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F\,F}\,I^{-1}M_{uta}\right)\underline{x}_a + \left(\frac{1}{m}F_{utb} - {}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F\,F}\,I^{-1}M_{utb}\right)\underline{x}_b \\ &+\left(-{}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F\,F}\,I^{-1}M_{urd}\right)\underline{x}_d + \left(-{}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F\,F}\,I^{-1}M_{ure}\right)\underline{x}_e + \left(\frac{1}{m}F_c - {}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F\,F}\,I^{-1}M_c\right)\underline{u} \\ &-{}^{(S)}\underline{a}_{in} + {}^{(S)}\underline{a}_d + \frac{1}{m}{}^{(S)}\underline{F}_b - {}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F^*F_u}\;{}^{(F)}\underline{\alpha}_d - {}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F^*F_u}\,I^{-1}\,{}^{(F)}\underline{M}_b \end{aligned} \qquad \text{(Eq. 89)} $$

and

$$\underline{\dot{x}}_e = \frac{1}{2}\,{}^{S/F}Q\, I^{-1}\left[M_{uta}\,\underline{x}_a + M_{utb}\,\underline{x}_b + M_{urd}\,\underline{x}_d + M_{ure}\,\underline{x}_e\right] + \frac{1}{2}\,{}^{S/F}Q\, I^{-1} M_c\,\underline{u} + \frac{1}{2}\,{}^{S/F}Q\;{}^{(F)}\underline{\alpha}_d + \frac{1}{2}\,{}^{S/F}Q\, I^{-1}\;{}^{(F)}\underline{M}_b . \qquad \text{(Eq. 90)}$$

For completeness, Equation (37) can be rewritten as $\underline{\dot{x}}_d = \underline{x}_e$. (Eq. 91)

To include ${}^{(S)}\underline{\ddot{r}}_{N_0E}$ as states, define $\underline{x}_c$ by

$$\omega_h\,{}^{(S)}\underline{\ddot{r}}_{N_0E} = \underline{\dot{x}}_c + \omega_h\,\underline{x}_c , \qquad \text{(Eq. 92)}$$

for some high value of circular frequency ω_h. Taking the Laplace Transform,

$$\mathcal{L}\left\{{}^{(S)}\underline{r}_{N_0E}\right\} = \left(\frac{s+\omega_h}{\omega_h s^2}\right)\mathcal{L}\left\{\underline{x}_c\right\}, \qquad \text{(Eq. 93)}$$

so that

$$\underline{x}_c \approx {}^{(S)}\underline{\ddot{r}}_{N_0E} \text{ for } \omega << \omega_h . \qquad \text{(Eq. 94)}$$

Now using Equations (78), (79), 83), (85), (86), and (92) with (49),

$$\begin{aligned}\underline{\dot{x}}_c = {} & \omega_h\left(\frac{1}{m}F_{uta} - {}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F^*E} I^{-1} M_{uta}\right)\underline{x}_a + \omega_h\left(\frac{1}{m}F_{utb} - {}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F^*E} I^{-1} M_{utb}\right)\underline{x}_b - \omega_h\underline{x}_c \\ & - \omega_h\left({}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F\,E} I^{-1} M_{urd}\right)\underline{x}_d - \omega_h\left({}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F\,E} I^{-1} M_{ure}\right)\underline{x}_e + \omega_h\left(\frac{1}{m}F_c - {}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F\,E} I^{-1} M_c\right)\underline{u} \\ & + \omega_h\,{}^{(S)}\underline{a}_d + \omega_h\left(\frac{1}{m}\right){}^{(S)}\underline{F}_b - \omega_h\left({}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F\,E}\right){}^{(F)}\underline{\alpha}_d - \omega_h\left({}^{S/F}Q\;{}^{(F)}\underline{r}^{\times}_{F\,E} I^{-1}\right){}^{(F)}\underline{M}_b .\end{aligned} \qquad \text{(Eq. 95)}$$

A state-space representation of the system is given by Equations (62), (89), (90), (91), and (95), for state vector

$$\underline{x} = \begin{Bmatrix}\underline{x}_a \\ \underline{x}_b \\ \underline{x}_c \\ \underline{x}_d \\ \underline{x}_e\end{Bmatrix} . \qquad \text{(Eq. 96)}$$

For the small rotation angles associated with the MIM, ${}^{S/F}Q$ is approximately equal to the 3×3 identity matrix, in which case the state equations have constant coefficients. Specifically,

$$\dot{\underline{x}}_a = \underline{x}_b, \tag{Eq. 97}$$

$$\begin{aligned}\dot{\underline{x}}_b =& \left(\frac{1}{m}F_{uta} - {}^{(F)}\underline{r}^{\times}_{F\,F}\, I^{-1} M_{uta}\right)\underline{x}_a + \left(\frac{1}{m}F_{utb} - {}^{(F)}\underline{r}^{\times}_{F\,F}\, I^{-1} M_{utb}\right)\underline{x}_b \\ & -\left({}^{(F)}\underline{r}^{\times}_{F\,F}\, I^{-1} M_{urd}\right)\underline{x}_d - \left({}^{(F)}\underline{r}^{\times}_{F\,F}\, I^{-1} M_{ure}\right)\underline{x}_e + \left(\frac{1}{m}F_c - {}^{(F)}\underline{r}^{\times}_{F\,F}\, I^{-1} M_c\right)\underline{u} \\ & - {}^{(S)}\underline{a}_{in} + {}^{(S)}\underline{a}_d + \frac{1}{m}\,{}^{(S)}\underline{F}_b - {}^{(F)}\underline{r}^{\times}_{F\,F}\,{}^{(F)}\underline{\alpha}_d - {}^{(F)}\underline{r}^{\times}_{F\,F}\, I^{-1}\,{}^{(F)}\underline{M}_b\,,\end{aligned} \tag{Eq. 98}$$

$$\begin{aligned}\dot{\underline{x}}_c =& \,\omega_h\left(\frac{1}{m}F_{uta} - {}^{(F)}\underline{r}^{\times}_{F\,E} I^{-1} M_{uta}\right)\underline{x}_a + \omega_h\left(\frac{1}{m}F_{utb} - {}^{(F)}\underline{r}^{\times}_{F\,E} I^{-1} M_{utb}\right)\underline{x}_b - \omega_h \underline{x}_c \\ & - \omega_h\left({}^{(F)}\underline{r}^{\times}_{F\,E} I^{-1} M_{urd}\right)\underline{x}_d - \omega_h\left({}^{(F)}\underline{r}^{\times}_{F\,E} I^{-1} M_{ure}\right)\underline{x}_e + \omega_h\left(\frac{1}{m}F_c - {}^{(F)}\underline{r}^{\times}_{F\,E} I^{-1} M_c\right)\underline{u} \\ & + \omega_h\,{}^{(S)}\underline{a}_d + \omega_h\left(\frac{1}{m}\right){}^{(S)}\underline{F}_b - \omega_h\,{}^{(F)}\underline{r}^{\times}_{F^*E}\left({}^{(F)}\underline{\alpha}_d + I^{-1}\,{}^{(F)}\underline{M}_b\right),\end{aligned} \tag{Eq. 99}$$

$$\dot{\underline{x}}_d = \underline{x}_e\,, \tag{Eq. 100}$$

$$\begin{aligned}\dot{\underline{x}}_e =& \frac{1}{2} I^{-1}\left[M_{uta}\underline{x}_a + M_{utb}\underline{x}_b + M_{urd}\underline{x}_d + M_{ure}\underline{x}_e\right] \\ & + \frac{1}{2} I^{-1} M_c\,\underline{u} + \frac{1}{2}\left({}^{(F)}\underline{\alpha}_d + I^{-1}\,{}^{(F)}\underline{M}_b\right).\end{aligned} \tag{Eq. 101}$$

State-Space Equations With MIM-2 States

The above state equations must be modified to account for the states actually used in MIM-2. Designate by the post-superscript R the states defined above; and by the post-superscript C the states used with MIM-2. The former set of states are as follows, for $i = 1, 2, 3$:

$$x^R_{ai} = \underline{r}_{F_{uh}F_u} \cdot \hat{\underline{s}}_i\,, \tag{Eq. 102}$$

$$x^R_{bi} = \dot{x}^R_{ai}\,, \tag{Eq. 103}$$

$$\underline{x}_c^R = \mathcal{L}^{-1}\left\{\left(\frac{\omega_h}{s+\omega_h}\right)\mathcal{L}\left({}^{(S)}\ddot{\underline{r}}_{N_0 E}\right)\right\}, \qquad \text{(Eq. 104)}$$

$$x_{di}^R = {}^{F/S}\underline{\beta}\cdot\hat{\underline{s}}_i\,, \qquad \text{(Eq. 105)}$$

and

$$x_{ei}^R = \dot{x}_{di}^R; \qquad \text{(Eq. 106)}$$

and the latter,

$$\underline{x}_a^C = {}^{(S)}\underline{r}_{F_h^* F^*}\,, \qquad \text{(Eq. 107)}$$

$$\underline{x}_b^C = \dot{\underline{x}}_a^C\,, \qquad \text{(Eq. 108)}$$

$$\underline{x}_c^C = \underline{x}_c^R = \mathcal{L}^{-1}\left\{\left(\frac{\omega_h}{s+\omega_h}\right)\mathcal{L}\left({}^{(S)}\ddot{\underline{r}}_{N_0 E}\right)\right\}, \qquad \text{(Eq. 109)}$$

$$\underline{x}_d^C = \underline{x}_d^R = {}^{F/S}\underline{\beta}\cdot\hat{\underline{s}}_i\,, \qquad \text{(Eq. 110)}$$

and

$$\underline{x}_e^C = \underline{x}_e^R = \dot{\underline{x}}_d^C\,. \qquad \text{(Eq. 111)}$$

Consider now the equation,

$$\underline{r}_{S\ F} = \underline{r}_{S\ F} + \underline{r}_{F\ F} + \underline{r}_{F\ F}\,. \qquad \text{(Eq. 112)}$$

Differentiating twice, under the prior assumption that ${}^{N}\underline{\omega}^{S}$ is negligible, leads to the following two equations:

$$\frac{{}^{S}d\underline{r}_{S\ F}}{dt} = \frac{{}^{S}d\underline{r}_{F^* F^*}}{dt} + {}^{S}\underline{\omega}^{F}\times\underline{r}_{F^* F_u} \qquad \text{(Eq. 113)}$$

and

$$\frac{{}^{S}d^2\underline{r}_{S_u F_u}}{dt^2} = \frac{{}^{S}d^2\underline{r}_{F_h^* F^*}}{dt^2} + {}^{S}\underline{\alpha}^{F}\times\underline{r}_{F\ F} + {}^{S}\underline{\omega}^{F}\times\left({}^{S}\underline{\omega}^{F}\times\underline{r}_{F\ F}\right). \qquad \text{(Eq. 114)}$$

Linearizing about ${}^{N}\underline{\omega}^{F} = \underline{0}$ as before, the following interrelationships are found to hold for the two sets of states:

$$\underline{x}_a^R = \underline{x}_a^C + \left({}^{S/F}Q\ {}^{(F)}\underline{r}_{F^* F} + {}^{(S)}\underline{r}_{F\ F^*}\right), \qquad \text{(Eq. 115)}$$

$$\underline{x}_b^R = \underline{x}_b^C - 2\ {}^{S/F}Q\ {}^{(F)}\underline{r}_{F^* F_u}^{\times}\ {}^{S/F}Q^T\ \underline{x}_e^R\,, \qquad \text{(Eq. 116)}$$

$$\dot{\underline{x}}_b^R = \dot{\underline{x}}_b^C - {}^{S/F}Q\ {}^{(F)}\underline{r}_{F^* F_u}^{\times} I^{-1}\ {}^{(F)}\underline{M}\,. \qquad \text{(Eq. 117)}$$

Designate by ${}^{(S)}\underline{r}_b$ the final two terms of Equation (115). Substituting now from Equations (115), (116), and (117) into Equations (62), (89), (90), (91), and (95) yields the following state equations for MIM-2:

$$\dot{\underline{x}}_a^C = \underline{x}_b^C, \tag{Eq. 118}$$

$$\begin{aligned} \dot{\underline{x}}_b^C =& \left(\frac{1}{m}F_{uta}\right)\underline{x}_a^C + \left(\frac{1}{m}F_{utb}\right)\underline{x}_b^C + \left(-\frac{2}{m}F_{utb}\,{}^{S/F}Q\,{}^{(F)}\underline{r}_{F^*F_u}^{\times}\,{}^{S/F}Q^T\right)\underline{x}_e^C \\ &+ \left(\frac{1}{m}F_c\right)\underline{u} - {}^{(S)}\underline{a}_{in} + {}^{(S)}\underline{a}_d + \left(\frac{1}{m}F_{uta}\right){}^{(S)}\underline{r}_b + \left(\frac{1}{m}\right){}^{(S)}\underline{F}_b, \end{aligned} \tag{Eq. 119}$$

$$\begin{aligned} \dot{\underline{x}}_c^C =& \omega_h\left(\frac{1}{m}F_{uta} - {}^{S/F}Q\,{}^{(F)}\underline{r}_{F\,E}^{\times}I^{-1}M_{uta}\right)\underline{x}_a^C + \omega_h\left(\frac{1}{m}F_{utb} - {}^{S/F}Q\,{}^{(F)}\underline{r}_{F\,E}^{\times}I^{-1}M_{utb}\right)\underline{x}_b^C \\ &- \omega_h\underline{x}_c^C + \omega_h\left(-{}^{S/F}Q\,{}^{(F)}\underline{r}_{F^*E}^{\times}I^{-1}M_{urd}\right)\underline{x}_d^C \\ &+ \omega_h\left(\frac{-2}{m}F_{utb}\,{}^{S/F}Q\,{}^{(F)}\underline{r}_{F\,F_u}^{\times}\,{}^{S/F}Q^T + 2\,{}^{S/F}Q\,{}^{(F)}\underline{r}_{F\,E}^{\times}I^{-1}M_{utb}\,{}^{S/F}Q\,{}^{(F)}\underline{r}_{F\,F_u}^{\times}\,{}^{S/F}Q^T - {}^{S/F}Q\,{}^{(F)}\underline{r}_{F^*E}^{\times}I^{-1}M_{ure}\right)\underline{x}_e^C \\ &+ \omega_h\left(\frac{1}{m}F_c - {}^{S/F}Q\,{}^{(F)}\underline{r}_{F^*E}^{\times}I^{-1}M_c\right)\underline{u} + \omega_h\,{}^{(S)}\underline{a}_d + \omega_h\left(\frac{1}{m}\right){}^{(S)}\underline{F}_b \\ &+ \omega_h\left(\frac{1}{m}F_{uta} - {}^{S/F}Q\,{}^{(F)}\underline{r}_{F\,E}^{\times}I^{-1}M_{uta}\right){}^{(S)}\underline{r}_b + \omega_h\left(-{}^{S/F}Q\,{}^{(F)}\underline{r}_{F\,E}^{\times}\right)\left({}^{(F)}\underline{\alpha}_d + I^{-1}\,{}^{(F)}\underline{M}_b\right), \end{aligned} \tag{Eq. 120}$$

$$\dot{\underline{x}}_d^C = \underline{x}_e^C, \tag{Eq. 121}$$

$$\begin{aligned} \dot{\underline{x}}_e^C =& \left(\frac{1}{2}\,{}^{S/F}Q\,I^{-1}M_{uta}\right)\underline{x}_a^C + \left(\frac{1}{2}\,{}^{S/F}Q\,I^{-1}M_{utb}\right)\underline{x}_b^C + \left(\frac{1}{2}\,{}^{S/F}Q\,I^{-1}M_{urd}\right)\underline{x}_d^C \\ &+ \left(\frac{1}{2}\,{}^{S/F}Q\,I^{-1}M_{ure} - {}^{S/F}Q\,I^{-1}M_{utb}\,{}^{S/F}Q\,{}^{(F)}\underline{r}_{F^*F_u}^{\times}\,{}^{S/F}Q^T\right)\underline{x}_e^C + \left(\frac{1}{2}\,{}^{S/F}Q\,I^{-1}M_c\right)\underline{u} \\ &+ \left(\frac{1}{2}\,{}^{S/F}Q\,I^{-1}M_{uta}\right){}^{(S)}\underline{r}_b + \left(\frac{1}{2}\,{}^{S/F}Q\right)\left({}^{(F)}\underline{\alpha}_d + I^{-1}\,{}^{(F)}\underline{M}_b\right). \end{aligned} \tag{Eq. 122}$$

Again, for the small rotation angles associated with the MIM, ${}^{S/F}Q$ is approximately equal to the 3×3 identity matrix; Equations (118) through (122) reduce to the following:

$$\dot{\underline{x}}_a^C = \underline{x}_b^C, \tag{Eq. 123}$$

$$\dot{\underline{x}}_b^C = \left(\frac{1}{m}F_{uta}\right)\underline{x}_a^C + \left(\frac{1}{m}F_{utb}\right)\underline{x}_b^C + \left(-\frac{2}{m}F_{utb}\,{}^{(F)}\underline{r}^{\times}_{F^*F_u}\right)\underline{x}_e^C + \left(\frac{1}{m}F_c\right)\underline{u}$$
$$- {}^{(S)}\underline{a}_{in} + {}^{(S)}\underline{a}_d + \left(\frac{1}{m}F_{uta}\right){}^{(S)}\underline{r}_b + \left(\frac{1}{m}\right){}^{(S)}\underline{F}_b\,, \qquad \text{(Eq. 124)}$$

$$\dot{\underline{x}}_c^C = \omega_h\left(\frac{1}{m}F_{uta} - {}^{(F)}\underline{r}^{\times}_{F^*E}I^{-1}M_{uta}\right)\underline{x}_a^C + \omega_h\left(\frac{1}{m}F_{utb} - {}^{(F)}\underline{r}^{\times}_{F^*E}I^{-1}M_{utb}\right)\underline{x}_b^C - \omega_h\underline{x}_c^C$$
$$+\omega_h\left(-{}^{(F)}\underline{r}^{\times}_{F^*E}I^{-1}M_{urd}\right)\underline{x}_d^C + \omega_h\left(\frac{-2}{m}F_{utb}\,{}^{(F)}\underline{r}^{\times}_{F^*F_u} + 2\,{}^{(F)}\underline{r}^{\times}_{F^*E}I^{-1}M_{utb}\,{}^{(F)}\underline{r}^{\times}_{F^*F_u} - {}^{(F)}\underline{r}^{\times}_{F^*E}I^{-1}M_{ure}\right)\underline{x}_e^C \qquad \text{(Eq. 125)}$$
$$+\omega_h\left(\frac{1}{m}F_c - {}^{(F)}\underline{r}^{\times}_{F^*E}I^{-1}M_c\right)\underline{u} + \omega_h\,{}^{(S)}\underline{a}_d + \omega_h\left(\frac{1}{m}\right){}^{(S)}\underline{F}_b + \omega_h\left(\frac{1}{m}F_{uta} - {}^{(F)}\underline{r}^{\times}_{F^*E}I^{-1}M_{uta}\right){}^{(S)}\underline{r}_b$$
$$+\omega_h\left(-{}^{(F)}\underline{r}^{\times}_{F^*E}\right)\left({}^{(F)}\underline{\alpha}_d + I^{-1}\,{}^{(F)}\underline{M}_b\right),$$

$$\dot{\underline{x}}_d^C = \underline{x}_e^C\,, \qquad \text{(Eq. 126)}$$

$$\dot{\underline{x}}_e^C = \left(\frac{1}{2}I^{-1}M_{uta}\right)\underline{x}_a^C + \left(\frac{1}{2}I^{-1}M_{utb}\right)\underline{x}_b^C + \left(\frac{1}{2}I^{-1}M_{urd}\right)\underline{x}_d^C$$
$$+\left(\frac{1}{2}I^{-1}M_{ure} - I^{-1}M_{utb}\,{}^{(F)}\underline{r}^{\times}_{F^*F_u}\right)\underline{x}_e^C + \left(\frac{1}{2}I^{-1}M_c\right)\underline{u} \qquad \text{(Eq. 127)}$$
$$+\left(\frac{1}{2}I^{-1}M_{uta}\right){}^{(S)}\underline{r}_b + \left(\frac{1}{2}\right)\left({}^{(F)}\underline{\alpha}_d + I^{-1}\,{}^{(F)}\underline{M}_b\right).$$

State-Space Equations Using Kane's Dynamics [10]

The above state equations for MIM-2 can be derived alternatively, by the approach commonly called "Kane's Dynamics." First define generalized coordinates q_i and generalized speeds u_i as follows. [Note the use of post-superscripts now, instead of the previous post-subscripts, with position (and, later, velocity and acceleration) vectors. The post-subscript position, with vectors, is used for another purpose in Kane's notation, as will be seen.]

$$q_1 = \underline{r}^{F_h^*F^*} \cdot \hat{\underline{s}}_1\,, \qquad \text{(Eq. 128)}$$

$$q_2 = \underline{r}^{F_h^*F^*} \cdot \hat{\underline{s}}_2\,, \qquad \text{(Eq. 129)}$$

$$q_3 = \underline{r}^{F_h^*F^*} \cdot \hat{\underline{s}}_3\,, \qquad \text{(Eq. 130)}$$

$$u_4 = {}^N\underline{\omega}^F \cdot \hat{\underline{s}}_1 = \dot{q}_4, \qquad \text{(Eq. 131)}$$

$$u_5 = {}^N\underline{\omega}^F \cdot \hat{\underline{s}}_2 = \dot{q}_5, \qquad \text{(Eq. 132)}$$

$$u_6 = {}^N\underline{\omega}^F \cdot \hat{\underline{s}}_3 = \dot{q}_6, \qquad \text{(Eq. 133)}$$

$$u_7 = \dot{q}_7 = -\omega_h q_7 + \omega_h \left(\ddot{\underline{r}}^{N_0E} \cdot \hat{\underline{s}}_1\right), \qquad \text{(Eq. 134)}$$

$$u_8 = \dot{q}_8 = -\omega_h q_8 + \omega_h \left(\ddot{\underline{r}}^{N_0E} \cdot \hat{\underline{s}}_2\right), \qquad \text{(Eq. 135)}$$

$$u_9 = \dot{q}_9 = -\omega_h q_9 + \omega_h \left(\ddot{\underline{r}}^{N_0E} \cdot \hat{\underline{s}}_3\right), \qquad \text{(Eq. 136)}$$

$$u_{10} = {}^N\underline{\omega}^S \cdot \hat{\underline{s}}_1 = \dot{q}_{10}, \qquad \text{(Eq. 137)}$$

$$u_{11} = {}^N\underline{\omega}^S \cdot \hat{\underline{s}}_2 = \dot{q}_{11}, \qquad \text{(Eq. 138)}$$

$$u_{12} = {}^N\underline{\omega}^S \cdot \hat{\underline{s}}_3 = \dot{q}_{12}, \qquad \text{(Eq. 139)}$$

$$u_1 = \dot{q}_1, \qquad \text{(Eq. 140)}$$

$$u_2 = \dot{q}_2, \qquad \text{(Eq. 141)}$$

and

$$u_3 = \dot{q}_3. \qquad \text{(Eq. 142)}$$

Next determine useful velocities and angular velocities in terms of these generalized coordinates and generalized speeds. The angular velocities can be expressed as follows:

$${}^N\underline{\omega}^S = u_{10}\hat{\underline{s}}_1 + u_{11}\hat{\underline{s}}_2 + u_{12}\hat{\underline{s}}_3, \qquad \text{(Eq. 143)}$$

$${}^N\underline{\omega}^F = u_4\hat{\underline{s}}_1 + u_5\hat{\underline{s}}_2 + u_6\hat{\underline{s}}_3, \qquad \text{(Eq. 144)}$$

and

$${}^S\underline{\omega}^F = {}^N\underline{\omega}^F - {}^N\underline{\omega}^S = \left(u_4 - u_{10}\right)\hat{\underline{s}}_1 + \left(u_5 - u_{11}\right)\hat{\underline{s}}_2 + \left(u_6 - u_{12}\right)\hat{\underline{s}}_3. \qquad \text{(Eq. 145)}$$

The velocity of the flotor center-of-mass is

$$\underline{v}^F = \frac{{}^N d}{dt}\left(\underline{r}^{N_0S} + \underline{r}^{S\,F} + \underline{r}^{F\,F}\right) = \underline{v}^S + {}^N\underline{\omega}^S \times \underline{r}^{S\,F} + \frac{{}^S d}{dt}\left(\underline{r}^{F\,F}\right) + {}^N\underline{\omega}^S \times \underline{r}^{F\,F}. \qquad \text{(Eq. 146)}$$

In terms of the generalized coordinates and generalized speeds,

$$\begin{aligned}\underline{v}^F &= \underline{v}^S + {}^N\underline{\omega}^S \times \underline{r}^{SF} + \underline{\hat{s}}_1\left(u_1 + q_3u_{11} - q_2u_{12}\right)\\ &\quad + \underline{\hat{s}}_2\left(u_2 + q_1u_{12} - q_3u_{10}\right) + \underline{\hat{s}}_3\left(u_3 + q_2u_{10} - q_1u_{11}\right).\end{aligned} \tag{Eq. 147}$$

Accordingly, the respective velocities of the isolation point E, the eight actuator force-application points B_i, and the umbilical-attachment point F_u, are as follows:

$$\begin{aligned}\underline{v}^E &= \underline{v}^S + {}^N\underline{\omega}^S \times \underline{r}^{SF^*} + \underline{\hat{s}}_1\left(u_1 + q_3u_{11} - q_2u_{12}\right)\\ &\quad + \underline{\hat{s}}_2\left(u_2 + q_1u_{12} - q_3u_{10}\right) + \underline{\hat{s}}_3\left(u_3 + q_2u_{10} - q_1u_{11}\right)\\ &\quad + {}^N\underline{\omega}^F \times \underline{r}^{F^*E},\end{aligned} \tag{Eq. 148}$$

$$\begin{aligned}\underline{v}^{B_i} &= \underline{v}^{S_u} + {}^N\underline{\omega}^S \times \underline{r}^{S_uF_h^*} + \underline{\hat{s}}_1\left(u_1 + q_3u_{11} - q_2u_{12}\right)\\ &\quad + \underline{\hat{s}}_2\left(u_2 + q_1u_{12} - q_3u_{10}\right) + \underline{\hat{s}}_3\left(u_3 + q_2u_{10} - q_1u_{11}\right)\\ &\quad + {}^N\underline{\omega}^F \times \underline{r}^{F^*B_i},\end{aligned} \tag{Eq. 149}$$

and

$$\begin{aligned}\underline{v}^{F_u} &= \underline{v}^{S_u} + {}^N\underline{\omega}^S \times \underline{r}^{S_uF_h^*} + \underline{\hat{s}}_1\left(u_1 + q_3u_{11} - q_2u_{12}\right)\\ &\quad + \underline{\hat{s}}_2\left(u_2 + q_1u_{12} - q_3u_{10}\right) + \underline{\hat{s}}_3\left(u_3 + q_2u_{10} - q_1u_{11}\right)\\ &\quad + {}^N\underline{\omega}^F \times \underline{r}^{F^*F_u}.\end{aligned} \tag{Eq. 150}$$

One can now express the linearized partial velocities (L.P.V.'s) and linearized partial angular velocities (L.P.A.V.'s), corresponding to the foregoing velocities and angular velocities, using the following notation: $\underline{v}_i^E = \dfrac{\partial\, \underline{v}^E}{\partial\, u_i}$ is the partial velocity (P.V.) of point E with respect to the i^{th} generalized speed u_i, ${}^N\underline{\omega}_i^F = \dfrac{\partial\, {}^N\underline{\omega}^F}{\partial\, u_i}$ is the partial angular velocity (P.A.V.) of reference frame F with respect to reference frame N, and ${}_l\underline{v}_i^E$ and ${}^N_l\underline{\omega}_i^F$ are the respective linearized velocity terms. Assume now that ${}^N\underline{\omega}^S \approx \underline{0}$, so that $u_{10} = u_{11} = u_{12} = 0$. Then $\underline{v}_i^{S_u} = \underline{0}$ $(i = 10,11,12)$. Assume additionally that the flotor mass is much smaller than that of the combined stator-plus-orbiter, in which case ${}^{S}\underline{\omega}^{F}$ does not affect $\underline{v}^{S_u}$. Then $\underline{v}_i^{S_u} = \underline{0}$ $(i = 4,5,6)$. Similarly, since $\underline{v}^{S_u}$ is not affected by $\underline{r}^{F_h^* F^*}$ or $\ddot{\underline{r}}^{N_0 E}$, $\underline{v}_i^{S_u} = \underline{0}$ $(i = 1,2,3,7,8,9)$. That is,

$$\underline{v}_i^{S_u} = \underline{0} \ (i = 1, \ldots, 12)\,. \qquad \text{(Eq. 151)}$$

Further, since $\underline{v}^{S_u}$ is unaffected by $\underline{r}^{F^*E}$, $\underline{r}^{F^*B_i}$, or $\underline{r}^{F^*F_u}$, one has the following L.P.V.'s and L.P.A.V.'s:

$$ {}_{l}\underline{v}_i^{F^*} = \hat{\underline{s}}_i \ (i = 1, 2, 3)\,, \qquad \text{(Eq. 152)}$$

$$ {}_{l}\underline{v}_i^{F^*} = \underline{0} \ (i = 4, \ldots, 9)\,, \qquad \text{(Eq. 153)}$$

$$ {}_{l}\underline{v}_i^{E} = {}_{l}\underline{v}_i^{B_j} = {}_{l}\underline{v}_i^{F_u} = {}_{l}\underline{v}_i^{F^*} \ \forall i \text{ in } \{1,\ldots,9\} \text{ and } \forall j \text{ in } \{1,\ldots,8\}, \qquad \text{(Eq. 154)}$$

$$ {}_{l}^{S}\underline{\omega}_i^{F} = \underline{0} \ (i = 1, 2, 3, 7, 8, 9), \qquad \text{(Eq. 155)}$$

$$ {}_{l}^{S}\underline{\omega}_4^{F} = \hat{\underline{s}}_1\,, \qquad \text{(Eq. 156)}$$

$$ {}_{l}^{S}\underline{\omega}_5^{F} = \hat{\underline{s}}_2\,, \qquad \text{(Eq. 157)}$$

$$ {}_{l}^{S}\underline{\omega}_6^{F} = \hat{\underline{s}}_3\,, \qquad \text{(Eq. 158)}$$

and

$$ {}_{l}^{N}\underline{\omega}_i^{F} = {}_{l}^{S}\underline{\omega}_i^{F} \ (i = 1, \ldots, 9)\,. \qquad \text{(Eq. 159)}$$

All L.P.V.'s and L.P.A.V.'s associated with u_{10}, u_{11}, and u_{12} are $\underline{0}$.

In order to determine the linearized accelerations (L.A.'s) and linearized angular accelerations (L.A.A.'s), one must first determine the linearized velocities (L.V.'s) and the linearized angular velocities (L.A.V.'s). Assuming still that ${}^{N}\underline{\omega}^{S} \approx \underline{0}$, the L.V.'s and L.A.V.'s are as follows:

$$ {}_{l}\underline{v}^{F^*} = \underline{v}^{S_u} + u_1\hat{\underline{s}}_1 + u_2\hat{\underline{s}}_2 + u_3\hat{\underline{s}}_3\,, \qquad \text{(Eq. 160)}$$

$$ {}_{l}\underline{v}^{E} = \underline{v}^{S_u} + u_1\hat{\underline{s}}_1 + u_2\hat{\underline{s}}_2 + u_3\hat{\underline{s}}_3 + {}_{l}\left({}^{S}\underline{\omega}^{F} \times \underline{r}^{F^*E}\right), \qquad \text{(Eq. 161)}$$

$$ {}_{l}\underline{v}^{B_i} = \underline{v}^{S_u} + u_1\underline{\hat{s}}_1 + u_2\underline{\hat{s}}_2 + u_3\underline{\hat{s}}_3 + {}_{l}\left({}^{S}\underline{\omega}^{F} \times \underline{r}^{F^*B_i}\right), \qquad \text{(Eq. 162)} $$

$$ {}_{l}\underline{v}^{F_u} = \underline{v}^{S_u} + u_1\underline{\hat{s}}_1 + u_2\underline{\hat{s}}_2 + u_3\underline{\hat{s}}_3 + {}_{l}\left({}^{S}\underline{\omega}^{F} \times \underline{r}^{F^*F_u}\right), \qquad \text{(Eq. 163)} $$

$$ {}_{l}^{N}\underline{\omega}^{S} = \underline{0}, \qquad \text{(Eq. 164)} $$

and

$$ {}_{l}^{N}\underline{\omega}^{F} = {}_{l}^{S}\underline{\omega}^{F} = u_4\underline{\hat{s}}_1 + u_5\underline{\hat{s}}_2 + u_6\underline{\hat{s}}_3. \qquad \text{(Eq. 165)} $$

Differentiating the L.V.'s and L.A.V.'s, and linearizing about ${}^{S}\underline{\omega}^{F} \approx \underline{0}$, yields the following L.A.'s and L.A.A.'s:

$$ {}_{l}\underline{a}^{F^*} = \underline{a}^{S_u} + \dot{u}_1\underline{\hat{s}}_1 + \dot{u}_2\underline{\hat{s}}_2 + \dot{u}_3\underline{\hat{s}}_3, \qquad \text{(Eq. 166)} $$

$$ {}_{l}\underline{a}^{E} = \underline{a}^{S_u} + \dot{u}_1\underline{\hat{s}}_1 + \dot{u}_2\underline{\hat{s}}_2 + \dot{u}_3\underline{\hat{s}}_3 + \left(\dot{u}_4\underline{\hat{s}}_1 + \dot{u}_5\underline{\hat{s}}_2 + \dot{u}_6\underline{\hat{s}}_3\right) \times \underline{r}^{F^*E}, \qquad \text{(Eq. 167)} $$

$$ {}_{l}\underline{a}^{B_i} = \underline{a}^{S_u} + \dot{u}_1\underline{\hat{s}}_1 + \dot{u}_2\underline{\hat{s}}_2 + \dot{u}_3\underline{\hat{s}}_3 + \left(\dot{u}_4\underline{\hat{s}}_1 + \dot{u}_5\underline{\hat{s}}_2 + \dot{u}_6\underline{\hat{s}}_3\right) \times \underline{r}^{F^*B_i}, \qquad \text{(Eq. 168)} $$

$$ {}_{l}\underline{a}^{F_u} = \underline{a}^{S_u} + \dot{u}_1\underline{\hat{s}}_1 + \dot{u}_2\underline{\hat{s}}_2 + \dot{u}_3\underline{\hat{s}}_3 + \left(\dot{u}_4\underline{\hat{s}}_1 + \dot{u}_5\underline{\hat{s}}_2 + \dot{u}_6\underline{\hat{s}}_3\right) \times \underline{r}^{F^*F_u}, \qquad \text{(Eq. 169)} $$

$$ {}_{l}^{N}\underline{\alpha}^{S} = \underline{0}, \qquad \text{(Eq. 170)} $$

and

$$ {}_{l}^{N}\underline{\alpha}^{F} = {}_{l}^{S}\underline{\alpha}^{F} = \dot{u}_4\underline{\hat{s}}_1 + \dot{u}_5\underline{\hat{s}}_2 + \dot{u}_6\underline{\hat{s}}_3. \qquad \text{(Eq. 171)} $$

Beginning with Equation (3), one can also obtain readily an alternate expression for ${}_{l}^{S}\underline{\alpha}^{F}$:

$$ {}_{l}^{S}\underline{\alpha}^{F} = \underline{\underline{I}}^{-1} \cdot \left\{ \underline{M}_{ur} + \underline{M}_{d} + \left[\underline{r}^{F\,F} \times \left(\underline{F}_{ut} - \underline{F}_{b}\right)\right] + \left(\sum_{i=1}^{8} \underline{r}^{F\,B} \times \underline{F}_{c}^{i}\right) \right\} \qquad \text{(Eq. 172)} $$

The final step, before writing the generalized active forces and generalized inertia forces of Kane's equations, is to determine the contributing loads:

The resultant of the actuator forces (cf. Eq. 44), which are considered to be applied at respective locations B_i, is

$$ \underline{F}_c = \sum_{i=1}^{8} \underline{F}_c^{i} = \sum_{i=1}^{8} \left(-I_i L_i B_i \underline{\hat{l}}_i \times \underline{\hat{B}}_i\right). \qquad \text{(Eq. 173)} $$

The umbilical force (cf. Eq. 15), with the former term (in curly brackets) applied at F_u, and the latter (umbilical bias force $\underline{F}_b$) at F^*, is

$$\underline{F}_{ut} = \sum_{i=1}^{3} \left\{ -K_t^i \left(\underline{r}^{F_{uh}F_u} \cdot \underline{\hat{s}}_i \right) \underline{\hat{s}}_i - C_t^i \left[\frac{{}^N d}{dt} \left(\underline{r}^{F_{uh}F_u} \right) \cdot \underline{\hat{s}}_i \right] \underline{\hat{s}}_i \right\} + \underline{F}_b . \qquad \text{(Eq. 174)}$$

The direct disturbance force (cf. Eq. 41), applied at F^*, is $\underline{F}_d$ (unknown).

The umbilical moment (cf. Eq. 20), applied about F^*, is

$$\underline{M}_{ur} = \sum_{i=1}^{3} \left\{ -K_r^i \left[\left(\phi^{F/S} \underline{\hat{n}}_\phi \right) \cdot \underline{\hat{s}}_i \right] \underline{\hat{s}}_i - C_r^i \left[\left(\dot{\phi}^{F/S} \underline{\hat{n}}_\phi \right) \cdot \underline{\hat{s}}_i \right] \underline{\hat{s}}_i \right\} + \underline{M}_b . \qquad \text{(Eq. 175)}$$

The direct disturbance moment (cf. Eq. 42), applied about F^*, is $\underline{M}_d$ (unknown). (Eq. 176)

The above expressions for the contributing loads can also be written in measure-number form, needed for eventual computer implementation. Equation (173) becomes

$${}^{(S)}\underline{F}_c = \left[-L_1 \, {}^{(S)}\underline{\hat{I}}_1^{\times} \, {}^{S/F}Q \, B_1 \, {}^{(F)}\underline{\hat{B}}_1 \right] I_1 + \ldots + \left[-L_8 \, {}^{(S)}\underline{\hat{I}}_8^{\times} \, {}^{S/F}Q \, B_8 \, {}^{(F)}\underline{\hat{B}}_8 \right] I_8 = F_c \, \underline{I} , \qquad \text{(Eq. 177a, b)}$$

where

$$\underline{I} = \left[I_1, \ldots, I_8 \right]^T . \qquad \text{(Eq. 178)}$$

Equation (174) is first re-expressed as

$$\underline{F}_{ut} = \sum_{i=1}^{3} \left\{ -K_t^i \left[\left(\underline{r}^{F_{uh}F_h^*} + \underline{r}^{F_h^*F^*} + \underline{r}^{F^*F_u} \right) \cdot \underline{\hat{s}}_i \right] \underline{\hat{s}}_i - C_t^i \left[\frac{{}^N d}{dt} \left(\underline{r}^{F_{uh}F_h^*} + \underline{r}^{F_h^*F^*} + \underline{r}^{F^*F_u} \right) \cdot \underline{\hat{s}}_i \right] \underline{\hat{s}}_i \right\} + \underline{F}_b . \qquad \text{(Eq. 179)}$$

This now can readily be rewritten as

$${}^{(S)}\underline{F}_{ut} = F_{uta} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + F_{utb} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + {}^{(S)}\underline{F}_b - F_{utb} \, {}^{S/F}Q \; {}^{(F)}\underline{r}^{F^*F_u \times} \, {}^{S/F}Q^T \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} + F_{uta} \, {}^{(S)}\underline{r}_b . \qquad \text{(Eq. 180)}$$

The direct disturbance force is simply ${}^{(S)}\underline{F}_d$.

Equation (175) becomes

$$ {}^{(F)}\underline{M}_{ur} = \tilde{M}_{urd}\begin{Bmatrix} q_4 \\ q_5 \\ q_6 \end{Bmatrix} + \tilde{M}_{ure}\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} + {}^{(F)}\underline{M}_b , \qquad \text{(Eq. 181)} $$

where $$ \tilde{M}_{urd} = -{}^{S/F}Q^T K_r \qquad \text{(Eq. 182)} $$

and $$ \tilde{M}_{ure} = -{}^{S/F}Q^T C_r . \qquad \text{(Eq. 183)} $$

The direct disturbance moment is ${}^{(F)}\underline{M}_d = {}^{(F)}\left(\underline{\underline{I}} \cdot \underline{\alpha}_d\right)$. (Eq. 184)

Using the expressions for the L.P.V.'s and the L.P.A.V.'s with the contributing loads, the generalized active forces Q_i can be determined as follows:

$$ \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} {}_I\underline{v}_1^{F^*} \cdot \left(\underline{F}_d + \underline{F}_b\right) \\ {}_I\underline{v}_2^{F^*} \cdot \left(\underline{F}_d + \underline{F}_b\right) \\ {}_I\underline{v}_3^{F^*} \cdot \left(\underline{F}_d + \underline{F}_b\right) \end{Bmatrix} + \begin{Bmatrix} {}_I\underline{v}_1^{F_u} \cdot \left(\underline{F}_{ut} - \underline{F}_b\right) \\ {}_I\underline{v}_2^{F_u} \cdot \left(\underline{F}_{ut} - \underline{F}_b\right) \\ {}_I\underline{v}_3^{F_u} \cdot \left(\underline{F}_{ut} - \underline{F}_b\right) \end{Bmatrix} + \sum_{i=1}^{8} \begin{Bmatrix} {}_I\underline{v}_1^{B_i} \cdot \underline{F}_c^{\,i} \\ {}_I\underline{v}_2^{B_i} \cdot \underline{F}_c^{\,i} \\ {}_I\underline{v}_3^{B_i} \cdot \underline{F}_c^{\,i} \end{Bmatrix} $$

$$ = {}^{(S)}\underline{F}_{ut} + {}^{(S)}\underline{F}_c + {}^{(S)}\underline{F}_d , \qquad \text{(Eq. 185a, b)} $$

$$ \begin{Bmatrix} Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} {}_I^S\underline{\omega}_4^F \cdot \left\langle \underline{M}_{ur} + \underline{M}_d + \left[\underline{r}^{F\,F} \times \left(\underline{F}_{ut} - \underline{F}_b\right)\right] + \left(\sum_{i=1}^{8} \underline{r}^{F\,B} \times \underline{F}_c^{\,i}\right)\right\rangle \\ {}_I^S\underline{\omega}_5^F \cdot \left\langle \underline{M}_{ur} + \underline{M}_d + \left[\underline{r}^{F\,F} \times \left(\underline{F}_{ut} - \underline{F}_b\right)\right] + \left(\sum_{i=1}^{8} \underline{r}^{F\,B} \times \underline{F}_c^{\,i}\right)\right\rangle \\ {}_I^S\underline{\omega}_6^F \cdot \left\langle \underline{M}_{ur} + \underline{M}_d + \left[\underline{r}^{F\,F} \times \left(\underline{F}_{ut} - \underline{F}_b\right)\right] + \left(\sum_{i=1}^{8} \underline{r}^{F\,B} \times \underline{F}_c^{\,i}\right)\right\rangle \end{Bmatrix} $$

$$ = {}^{(S)}\left[\underline{r}^{F^*F_u} \times \left(\underline{F}_{ut} - \underline{F}_b\right)\right] + {}^{(S)}\left(\sum_i \underline{r}^{F^*B_i} \times \underline{F}_c^{\,i}\right) + {}^{S/F}Q\left({}^{(F)}\underline{M}_{ur} + {}^{(F)}\underline{M}_d\right), \qquad \text{(Eq. 186a, b)} $$

and $$ \begin{Bmatrix} Q_7 \\ Q_8 \\ Q_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} . \qquad \text{(Eq. 187)} $$

The generalized inertia forces are

$$\begin{Bmatrix} Q_1^* \\ Q_2^* \\ Q_3^* \end{Bmatrix} = \begin{Bmatrix} {}_I\underline{v}_1^F \cdot \left(-m \; {}_I\underline{a}^F\right) \\ {}_I\underline{v}_2^F \cdot \left(-m \; {}_I\underline{a}^F\right) \\ {}_I\underline{v}_3^F \cdot \left(-m \; {}_I\underline{a}^F\right) \end{Bmatrix} = -m \; {}^{(S)}_{I}\underline{a}^{F^*}, \qquad \text{(Eq. 188a, b)}$$

$$\begin{Bmatrix} Q_4^* \\ Q_5^* \\ Q_6^* \end{Bmatrix} = \begin{Bmatrix} {}^S_I\underline{\omega}_4^F \cdot \left(-\underline{\underline{I}} \cdot {}^S_I\underline{\alpha}^F - {}^S_I\underline{\omega}^F \times \underline{\underline{I}} \cdot {}^S_I\underline{\omega}^F\right) \\ {}^S_I\underline{\omega}_5^F \cdot \left(-\underline{\underline{I}} \cdot {}^S_I\underline{\alpha}^F - {}^S_I\underline{\omega}^F \times \underline{\underline{I}} \cdot {}^S_I\underline{\omega}^F\right) \\ {}^S_I\underline{\omega}_6^F \cdot \left(-\underline{\underline{I}} \cdot {}^S_I\underline{\alpha}^F - {}^S_I\underline{\omega}^F \times \underline{\underline{I}} \cdot {}^S_I\underline{\omega}^F\right) \end{Bmatrix}$$

$$= {}^{S/F}Q^{(F)}\left(-\underline{\underline{I}} \cdot {}^S_I\underline{\alpha}^F\right) \quad \text{(linearizing about } {}^S_I\underline{\omega}^F = \underline{0}\text{)}, \qquad \text{(Eq. 189a, b)}$$

and

$$\begin{Bmatrix} Q_7^* \\ Q_8^* \\ Q_9^* \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \qquad \text{(Eq. 190)}$$

Kane's Dynamical Equations for MIM can now be written in the following, recognizable forms of Newton's Second Law:

$${}^{(S)}\underline{F}_{ut} + {}^{(S)}\underline{F}_c + {}^{(S)}\underline{F}_d - m \; {}^{(S)}_{I}\underline{a}^{F^*} = \underline{0}, \qquad \text{(Eq. 191)}$$

where [cf. Eq. (166)]

$${}^{(S)}_{I}\underline{a}^{F^*} = {}^{(S)}\underline{a}_{in} + \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix}; \qquad \text{(Eq. 192)}$$

and

$${}^{(S)}\left[\underline{r}^{FF} \times \left(\underline{F}_{ut} - \underline{F}_b\right)\right] + {}^{(S)}\left(\sum_i \underline{r}^{FB} \times \underline{F}_c^i\right) + {}^{S/F}Q^{(F)}\underline{M}_{ur} + {}^{S/F}Q^{(F)}\underline{M}_d - {}^{S/F}Q^{(F)}\left(\underline{\underline{I}} \cdot {}^S_I\underline{\alpha}^F\right) = \underline{0}, \qquad \text{(Eq. 193)}$$

where [cf. Eq. (171)]

$${}^{(S)S}_{\;\;I}\underline{\alpha}^F = \begin{Bmatrix} \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \end{Bmatrix}. \qquad \text{(Eq. 194)}$$

These equations must now be placed into a usable state-space form.

Define the following:

$$\underline{x}_a = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}, \qquad \text{(Eq. 195)}$$

$$\underline{x}_b = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = \dot{\underline{x}}_a\,, \qquad \text{(Eq. 196a, b, c)}$$

$$\dot{\underline{x}}_c = \begin{Bmatrix} u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} \dot{q}_7 \\ \dot{q}_8 \\ \dot{q}_9 \end{Bmatrix} = -\omega_h \begin{Bmatrix} q_7 \\ q_8 \\ q_9 \end{Bmatrix} + \omega_h \, {}^{(S)}{}_{I}\underline{a}^{E}$$

$$= -\omega_h \underline{x}_c + \omega_h {}^{(S)}\left({}_{I}\underline{a}^{F} - \underline{r}^{F\,E} \times {}_{I}^{S}\underline{\alpha}^{F} \right), \qquad \text{(Eq. 197a, b, c, d)}$$

$$\tilde{\underline{x}}_d = \begin{Bmatrix} q_4 \\ q_5 \\ q_6 \end{Bmatrix}, \qquad \text{(Eq. 198)}$$

$$\tilde{\underline{x}}_e = \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{Bmatrix} = \dot{\tilde{\underline{x}}}_d\,, \qquad \text{(Eq. 199)}$$

and $$\underline{u} = \underline{I}. \qquad \text{(Eq. 200)}$$

Note that Equations (196) and (199) express Kane's Kinematical Equations.

Using state-definition equations (195), (196), and (199); force equations (177) and (180); acceleration equation (192); and disturbance equations (40) and (41); Kane's Dynamical Equation (191) becomes

$$\begin{aligned} \dot{\underline{x}}_b = &\left(\frac{1}{m}F_{uta}\right)\underline{x}_a + \left(\frac{1}{m}F_{utb}\right)\underline{x}_b + \left(-\frac{1}{m}F_{utb}\, {}^{S/F}Q\, {}^{(F)}\underline{r}^{\times}_{F^*F_u}\, {}^{S/F}Q^T\right)\tilde{\underline{x}}_e + \left(\frac{1}{m}F_c\right)\underline{u} \\ &- {}^{(S)}\underline{a}_{in} + {}^{(S)}\underline{a}_d + \left(\frac{1}{m}\right){}^{(S)}\underline{F}_b + \left(\frac{1}{m}F_{uta}\right){}^{(S)}\underline{r}_b. \end{aligned} \qquad \text{(Eq. 201)}$$

Likewise, using state-definition equations (195), (196), (198), and (199); force equations (177) and (180); moment equations (82), (83), (181), and (184); angular acceleration equation (194); and disturbance equation (42); Kane's Dynamical Equation (193) becomes

$$\begin{aligned}\dot{\underline{x}}_e = & \left({}^{S/F}Q\, I^{-1} M_{uta}\right)\underline{x}_a + \left({}^{S/F}Q\, I^{-1} M_{utb}\right)\underline{x}_b + \left({}^{S/F}Q\, I^{-1} \tilde{M}_{urd}\right)\underline{\tilde{x}}_d \\ & + \left({}^{S/F}Q\, I^{-1} \tilde{M}_{ure} - {}^{S/F}Q\, I^{-1} M_{utb}\, {}^{S/F}Q\; {}^{(F)}\underline{r}^{\times}_{F^*F_u}\; {}^{S/F}Q^T\right)\underline{\tilde{x}}_e + \left({}^{S/F}Q\, I^{-1} M_c\right)\underline{u} \\ & + \left({}^{S/F}Q\, I^{-1} M_{uta}\right){}^{(S)}\underline{r}_b + \left({}^{S/F}Q\right)\left({}^{(F)}\underline{\alpha}_d + I^{-1}\, {}^{(F)}\underline{M}_b\right).\end{aligned} \qquad \text{Eq. (202)}$$

In terms of Euler parameters $\underline{x}_d := {}^{(S)F/S}\underline{\beta}$ [where ${}^{F/S}\underline{\beta}$ is defined by Eq. (26)], for small rotation angle ϕ and negligible angular velocity ${}^{N}\underline{\omega}^{S}$ the rotational states are

$$\underline{\tilde{x}}_d = 2\underline{x}_d, \qquad \text{(Eq. 203)}$$

and

$$\underline{\tilde{x}}_e = 2\underline{x}_e. \qquad \text{(Eq. 204)}$$

Kinematical equations (196) and (199), and dynamical equations (201) and (202), now reduce to the forms found previously, *viz*., Equations (123), (126), (124) and (127), respectively.

Concluding Remarks

This paper has presented the derivation of algebraic, state-space equations for the Canadian Space Agency's Microgravity Vibration Isolation Mount. The states employed include payload relative translational position $\left(\underline{x}_a^C\right)$and velocity $\left(\underline{x}_b^C\right)$, payload relative rotation $\left(\underline{x}_d^C\right)$ and rotation rate $\left(\underline{x}_e^C\right)$, and payload translational acceleration $\left(\underline{x}_c^C\right)$. Feedback of $\underline{x}_a^C$ corresponds to a change in effective umbilical translational stiffness, with the effective umbilical assumed to be attached at the flotor center of mass. Similarly, feedback of $\underline{x}_b^C$, $\underline{x}_d^C$, or $\underline{x}_e^C$ corresponds, respectively, to a change in translational damping, rotational stiffness, or rotational damping, for the same effective umbilical. Likewise, feedback of payload translational acceleration causes a change in effective payload mass. Thus, a cost functional which penalizes these states produces an intuitive effect on system effective stiffness, damping, and inertia values.

The acceleration states can be selected to pertain to any arbitrary point on the flotor. This allows an optimal controller to be developed which penalizes directly the acceleration of any significant point of interest, such as the location of a crystal in a crystal-growth experiment.

The equations have been put into state-space form so that the powerful controller-design methods of optimal control theory (e.g., H_2 synthesis, H_∞ synthesis, μ synthesis, mixed-μ synthesis, and μ analysis) can be used. References [11], [12], and [13] detail the H_2 optimal controller design approach used for MIM.

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R. David Hampton, Bjarni V. Truggvason, Jean DeCarufel, Miles A. Townsend and William O. Wagar

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R. David Hampton, University of Alabama at Huntsville, Huntsville, Alabama 35899; Bjarni V. Tryggvason and Jean DeCarufel, Canadian Space Agency, St. Hubert, Quebec, Canada J3Y8Y9; Miles A. Townsend, Department of Mechanical, Aerospace, and Nuclear Engineering, University of Virginia, Charlottesville, Virginia, 22903; and William O. Wagar, NASA Lewis Research Center. Responsible person, William O.Wagar, organization code 6727, (216) 433–3665.

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Vibration acceleration levels on large space platforms exceed the requirements of many space experiments. The Microgravity Vibration Isolation Mount (MIM) was built by the Canadian Space Agency to attenuate these disturbances to acceptable levels, and has been operational on the Russian Space Station Mir since May 1996. It has demonstrated good isolation performance and has supported several materials science experiments. The MIM uses Lorentz (voice-coil) magnetic actuators to levitate and isolate payloads at the individual experiment/sub-experiment (versus rack) level. Payload acceleration, relative position, and relative orientation (Euler-parameter) measurements are fed to a state-space controller. The controller, in turn, determines the actuator currents needed for effective experiment isolation. This paper presents the development of an algebraic, state-space model of the MIM, in a form suitable for optimal controller design. The equations are first derived using Newton's Second Law directly; then a second derivation (i.e., validation) of the same equations is provided, using Kane's approach.

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